The Cayley Isomorphism Problem

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Basic Definitions and Background

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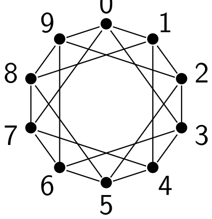

Figure: The Cayley graph $\mathrm{Cay}(\mathbb{Z}_{10}, \{1,3,7,9\})$.

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So $(x,y)\in E(\mathrm{Cay}(G,S))$ if and only if $x^{-1}y\in S$. This is sometimes used to define Cayley digraphs.

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Also, in order for $\mathrm{Cay}(G,S)$ to be loopless, it must be that $1 \notin S$.

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Two circulant graphs $\mathrm{Cay}(\mathbb{Z}_n, S)$ *and* $\mathrm{Cay}(\mathbb{Z}_n, T)$ *are isomorphic if and only if* $mS = \{ms : s \in S\} = T$ *for some* $m \in \mathbb{Z}_n^*$.

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The integer m is called a **multiplier**, and Ádám's conjecture is often stated as "two circulant graphs are isomorphic by a multiplier".

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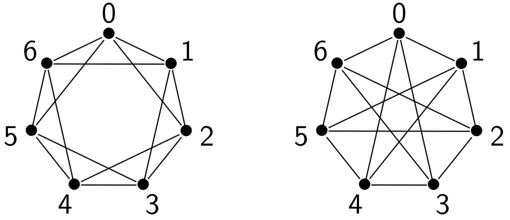

Figure: The Cayley graphs $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 5, 6\})$ and $\mathrm{Cay}(\mathbb{Z}_7, \{1, 3, 4, 6\})$

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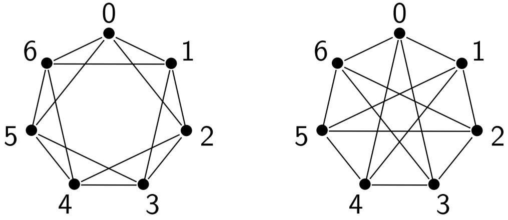

Figure: The Cayley graphs $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 5, 6\})$ and $\mathrm{Cay}(\mathbb{Z}_7, \{1, 3, 4, 6\})$

Note that $3 \cdot \{1, 2, 5, 6\} = \{1, 3, 4, 6\}$.

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This means that in checking whether or not two Cayley digraphs of G are isomorphic, one must always check the groups automorphisms of G.

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Problem 6

Determine the groups G for which any two isomorphic Cayley (di)graphs of G are isomorphic by a group automorphism of G.

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Let G be a group and $\mathrm{Cay}(G, S)$ a Cayley (di)graph of G. We say $\mathrm{Cay}(G, S)$ is a **CI-(di)graph of** *G if and only if whenever $\mathrm{Cay}(G, T)$ is another Cayley (di)graph of G then $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$ are isomorphic if and only they are isomorphic by a group automorphism of G.*

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So G is a CI-group with respect to (di)graphs if every Cayley (di)graph of G is a CI-(di)graph.

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A group $H \leq S_n$ is **transitive** *if whenever $x, y \in \mathbb{Z}_n$ (we assume here that S_n permutes the set $\mathbb{Z}_n$) then there exists $h \in H$ with $h(x) = y$.*

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Let G be a group and $g \in G$. Define $g_L : G \mapsto G$ by $g_L(x) = gx$. If $g_L(x) = g_L(y)$ then $gx = gy$ and $x = y$ so g_L is injective. Also, if $h \in G$ then $g_L(g^{-1}h) = gg^{-1}h = h$, so g_L is a bijection. If $(h, hs) \in E(\mathrm{Cay}(G, S))$ then $g_L(h, hs) = (gh, ghs) \in E(\mathrm{Cay}(G, S))$ and so $g_L \in \mathrm{Aut}(Cay(G, S))$.

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Sabidussi's Theorem shows us a way of defining when a combintorial object is a Cayley object of a group G:

Definition 14

A Cayley object X of G in some class $\mathcal{K}$ of combinatorial objects is a combinatorial object with $V(X) = G$ and $G_L \leq \mathrm{Aut}(X)$.

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Let $\mathcal{K}$ be a class of combinatorial objects. A group G for which any two isomorphic Cayley objects of G in $\mathcal{K}$ are isomorphic by a group automorphism of G is called a **CI-group with respect to** $\mathcal{K}$.

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So G is a CI-group with respect to $\mathcal{K}$ if every Cayley object of G in $\mathcal{K}$ is a CI-object.

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In reality the problem is even more general as we may also consider the problem for objects which have transitive automorphism groups but are not Cayley objects of any group G. Or just objects which have a large automorphism group that need not be transitive!

A general strategy to solve the isomorphism problem for digraphs with some symmetry

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Of course, if $\delta^{-1} g \delta \in \mathrm{Aut}(\Gamma)$ for every $g \in G$, then $\delta^{-1} G \delta \leq \mathrm{Aut}(\Gamma)$.

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So $\delta\gamma : \Gamma \mapsto \Delta$ is another isomorphism from Γ to Δ and $\gamma^{-1}\delta^{-1}G\delta\gamma \leq \mathrm{Aut}(\Gamma)$ by previous arguments. So the effect on $\delta^{-1}G\delta$ of replacing δ with $\gamma\delta$ is to conjugate $\delta^{-1}G\delta$ by γ.

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Lemma 19

$N_{S_G}(G_L) = \mathrm{Aut}(G) \cdot G_L$.

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What here is special about digraphs? Nothing! With almost identical proofs we have the following results.

Lemma 25 (Babai, 1977 [3])

Let G be a group, $\mathcal{K}$ a class of combinatorial objects, and X a Cayley object of G in $\mathcal{K}$.

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There does not seem to be a standard term for what is called here a complete block system of G. Other authors use a **system of imprimitivity** or a G-**invariant partition** for this term.

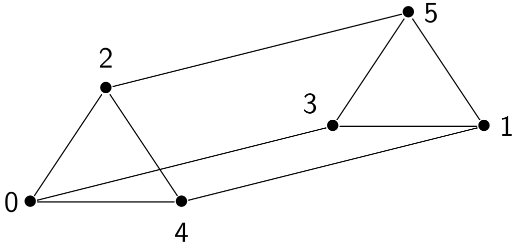
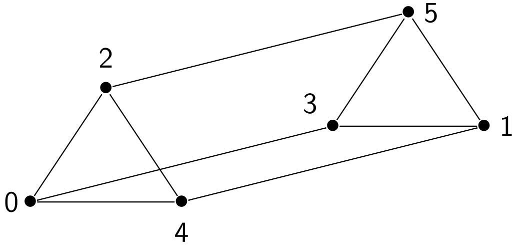


Figure: $\mathrm{Cay}(\mathbb{Z}_6, \{2, 3, 4\})$



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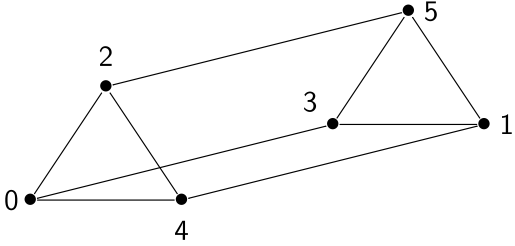

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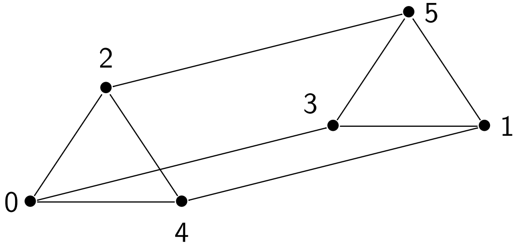

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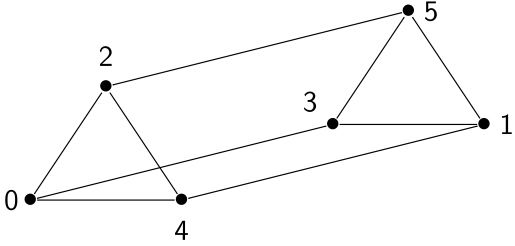

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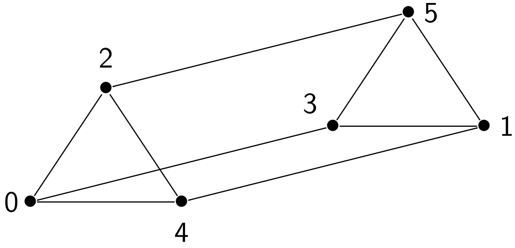

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Theorem 32

Let $\mathcal{B}$ be a complete block system of G. Then every block in $\mathcal{B}$ has the same cardinality, say k. Further, if m is the number of blocks in $\mathcal{B}$ then mk is the degree of G.

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Let G be a transitive group acting on X. If $N \triangleleft G$, then the orbits of N form a complete block system of G.

Proof: Let $x \in X$ and B the orbit of N that contains x, so $B = \{h(x) : h \in N\}$. Let $g \in G$, and for $h \in N$, denote by h' the element of N such that $gh = h'g$. Note h' always exists as $N \triangleleft G$, and that $\{h' : h \in N\} = N$ as conjugation by g induces an automorphism of N. Then

$$g(B) = \{gh(x) : h \in N\} = \{h'g(x) : h \in N\} = \{h(g(x)) : h \in N\}.$$

Hence $g(B)$ is the orbit of N that contains $g(x)$, and as the orbits of N form a partition of X, $g(B) \cap B = \emptyset$ or B. Thus B is a block, and as every conjugate block $g(B)$ of B is an orbit of N, the orbits of N do indeed form a complete block system of G.

Corollary 34

A transitive group G of prime degree p is primitive.

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The only divisors of p are 1 and p, and so the only possible blocks are singleton sets and the entire set on which G acts by Theorem 32. □

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A nontrivial normal subgroup of a primitive group is transitive.

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Let $G \leq S_n$ be primitive with $N \triangleleft G$.

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A nontrivial normal subgroup of a primitive group is transitive.

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Let $G \leq S_n$ be primitive with $N \triangleleft G$. Towards a contradiction, suppose that N is not transitive.

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A nontrivial normal subgroup of a primitive group is transitive.

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Let $G \leq S_n$ be primitive with $N \triangleleft G$. Towards a contradiction, suppose that N is not transitive. As the orbits of N form a complete block system $\mathcal{B}$ of G by Theorem 33, and, as N is nontrivial, $\mathcal{B}$ is nontrivial.

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Let $G \leq S_n$ be primitive with $N \triangleleft G$. Towards a contradiction, suppose that N is not transitive. As the orbits of N form a complete block system $\mathcal{B}$ of G by Theorem 33, and, as N is nontrivial, $\mathcal{B}$ is nontrivial. Thus G is imprimitive, contradicting Corollary 34. $\square$

Example: Define $\rho, \tau : \mathbb{Z}_2 \times \mathbb{Z}_5 \mapsto \mathbb{Z}_2 \times \mathbb{Z}_5$ by $\rho(i,j) = (i, j+1)$ and $\tau(i,j) = (i+1, 2j)$.

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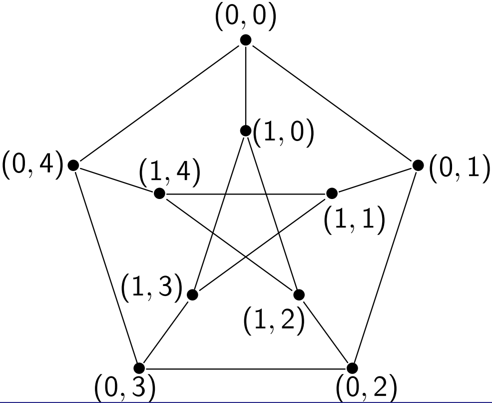

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and so $\langle\rho\rangle \triangleleft \langle\rho,\tau\rangle$. Then by Theorem 33 the orbits of $\langle\rho\rangle$, which are the sets $\{\{i,j\} : j \in \mathbb{Z}_5\} : i \in \mathbb{Z}_2\}$, form a complete block system of $\langle\rho,\tau\rangle$. Although we will not show this here, the full automorphism group of the Petersen graph is primitive.

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Our next goal is to show that every complete block system of a transitive group which contains a regular abelian subgroup is necessarily a normal complete block system. We will need several preliminary results.

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Let $G \leq S_n$ be transitive, and $x, y \in \mathbb{Z}_n$.

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We show that for $x \in \mathbb{Z}_n$ and $h \in G$, $\mathrm{Stab}_G(h(x)) = h\,\mathrm{Stab}_G(x)\,h^{-1}$.

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Corollary 37

A transitive abelian group $G \leq S_n$ is regular.

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Let $g \in G$ and $x \in \mathbb{Z}_n$. Then $g\mathrm{Stab}_G(x)g^{-1} = \mathrm{Stab}_G(g(x))$ by Corollary 36

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Let $g \in G$ and $x \in \mathbb{Z}_n$. Then $g\mathrm{Stab}_G(x)g^{-1} = \mathrm{Stab}_G(g(x))$ by Corollary 36 and as G is abelian,

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A transitive group $G \leq S_n$ is regular if and only if the order of G is the degree of G.

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Proof.

By the Orbit-Stabilizer Theorem (the size of an orbit G that contains x times the size of the stabilizer of x is the order of the group),

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Let $g \in G$ and $x \in \mathbb{Z}_n$. Then $g\mathrm{Stab}_G(x)g^{-1} = \mathrm{Stab}_G(g(x))$ by Corollary 36 and as G is abelian, $g\mathrm{Stab}_G(x)g^{-1} = \mathrm{Stab}_G(x)$. As G is transitive, $\mathrm{Stab}_G(x) = \mathrm{Stab}_G(y)$ for all $x, y \in \mathbb{Z}_n$. Then $\mathrm{Stab}_G(x) = 1$ and G is regular. $\square$

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By the Orbit-Stabilizer Theorem (the size of an orbit G that contains x times the size of the stabilizer of x is the order of the group), we see that $|G| = n \cdot |\mathrm{Stab}_G(x)| = n$. $\square$

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Definition 42

An equivalence relation $\equiv$ *as in the previous result is a* G-**congruence**.

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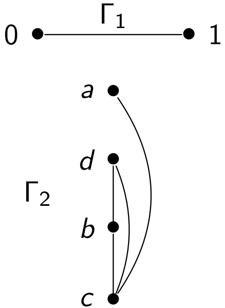

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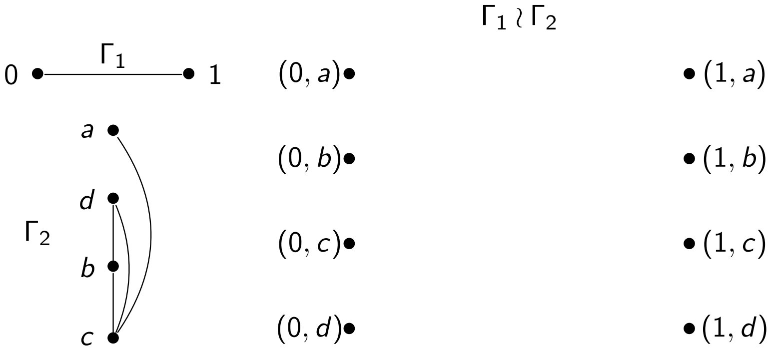

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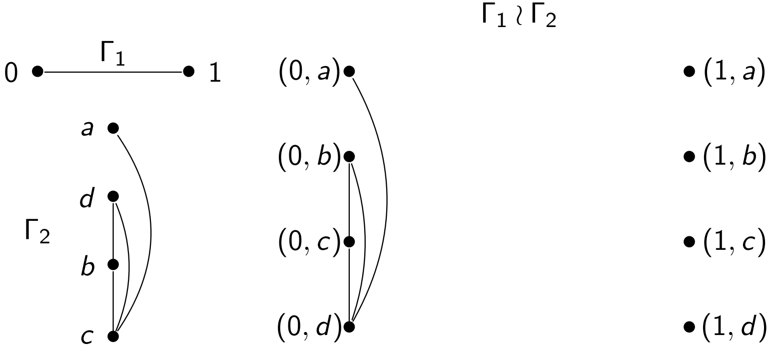

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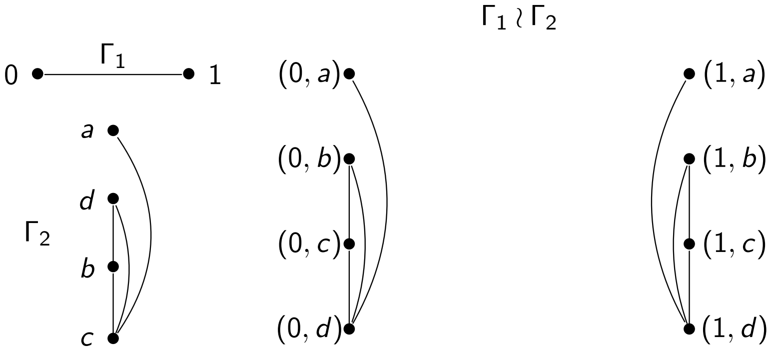

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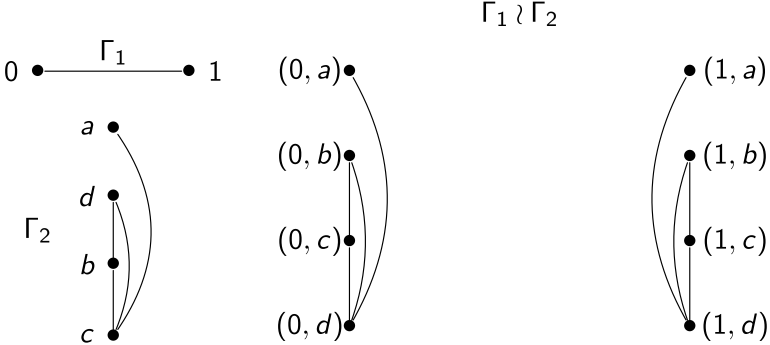

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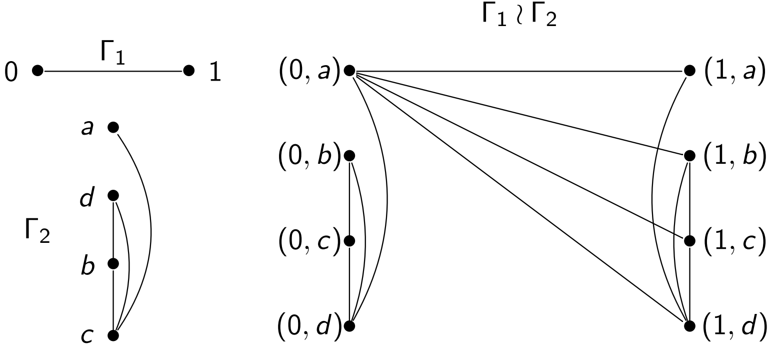

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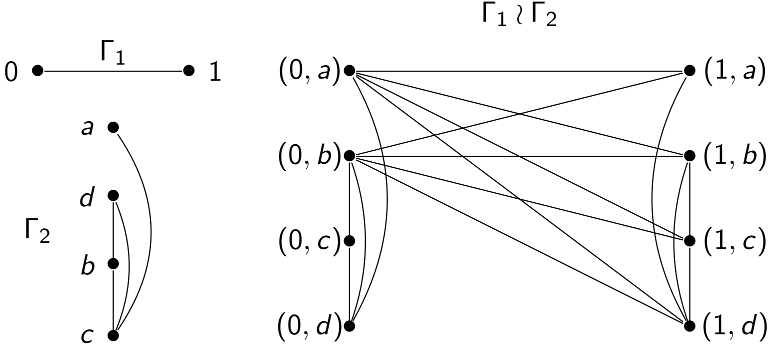

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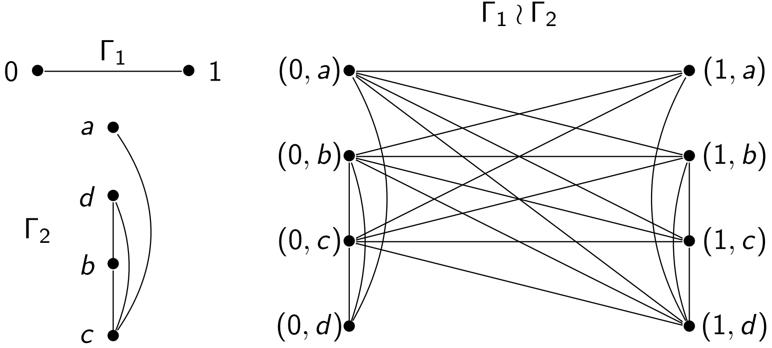

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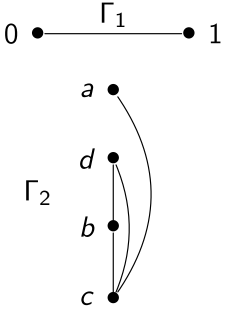

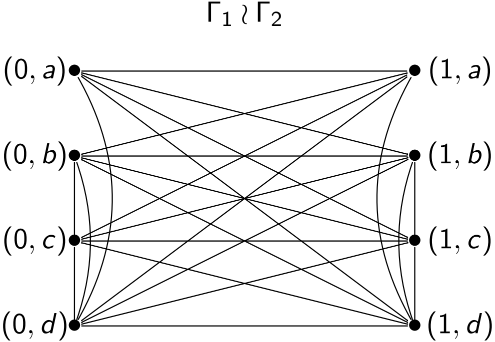

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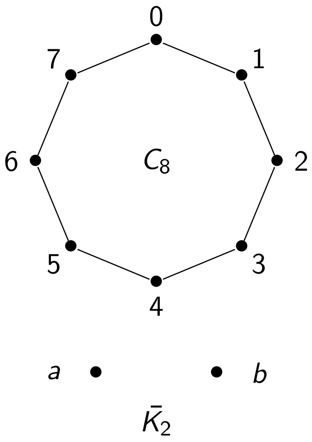

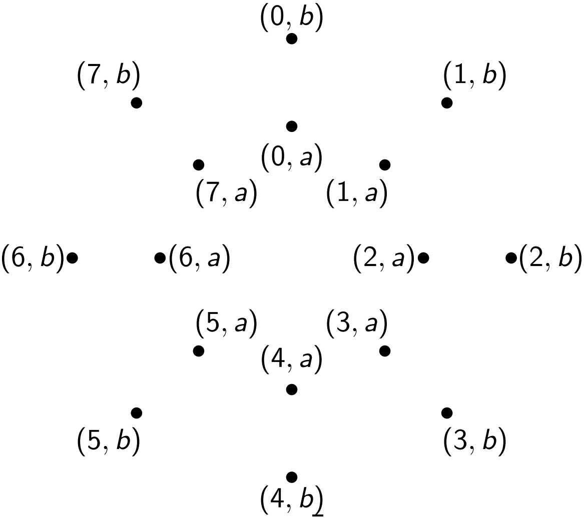

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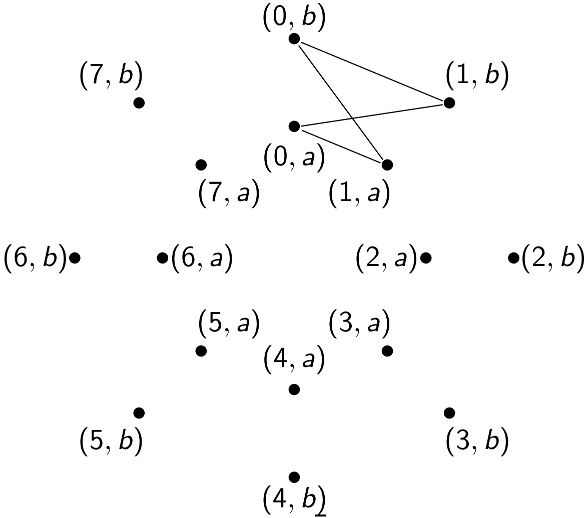

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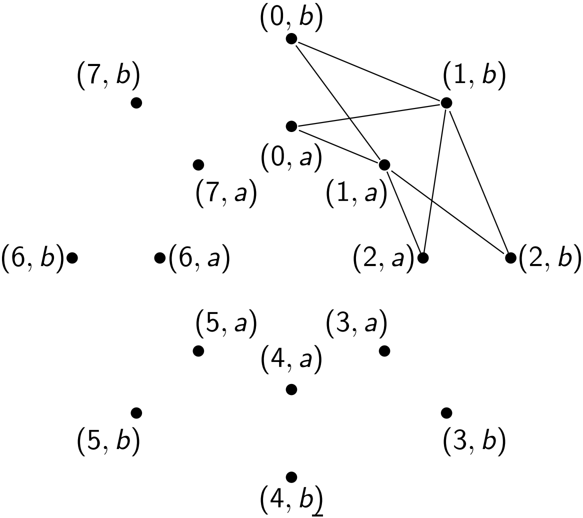

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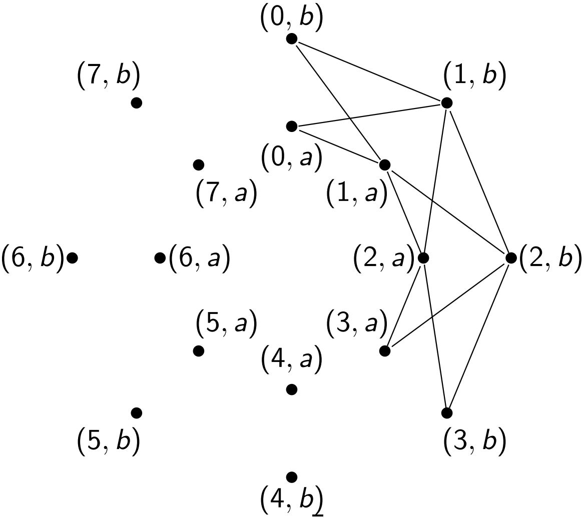

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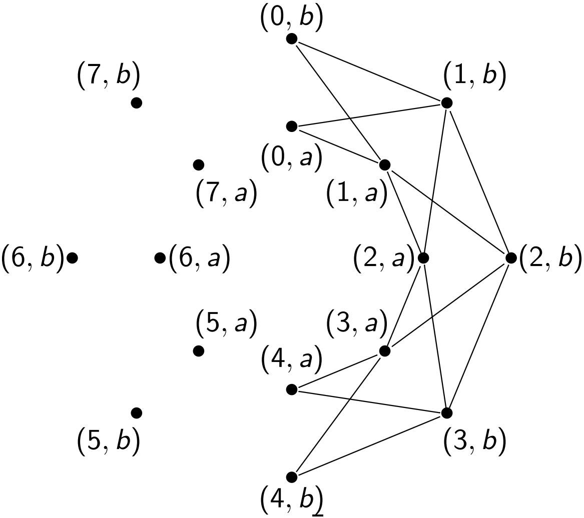

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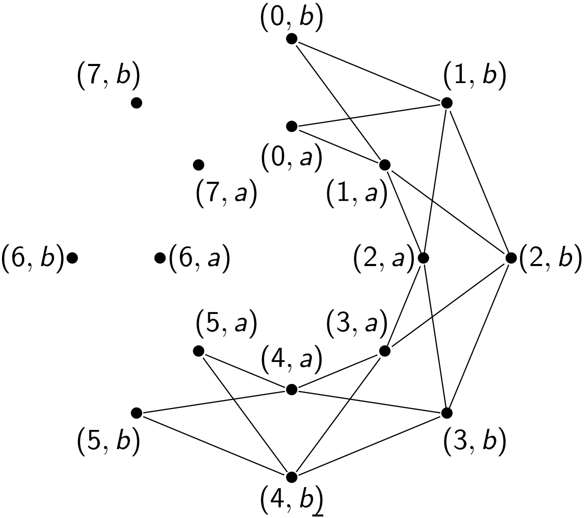

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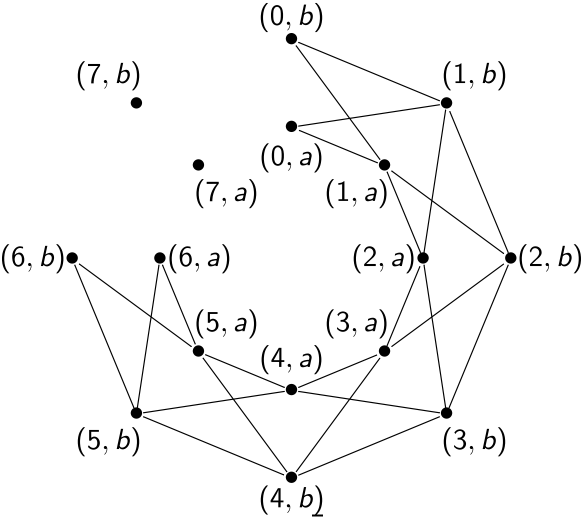

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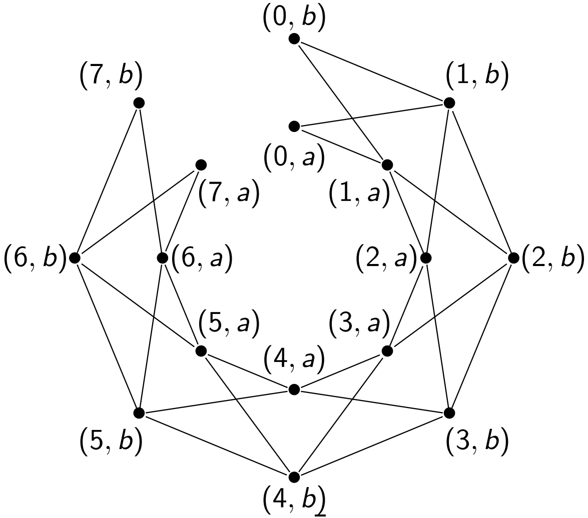

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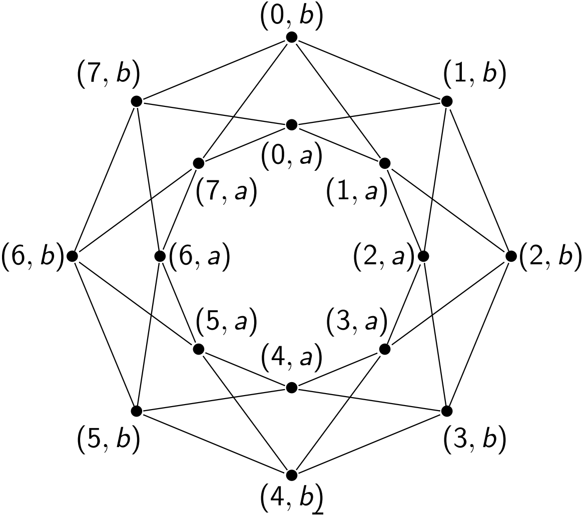

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It is easy to see that for digraphs Γ and Δ, $\mathrm{Aut}(\Gamma) \wr \mathrm{Aut}(\Delta) \leq \mathrm{Aut}(\Gamma \wr \Delta)$.

Example: The group $(\mathbb{Z}_p)_L \wr (\mathbb{Z}_p)_L = \{(i,j) \mapsto (i+a, j+b_i) : a, b_i \in \mathbb{Z}_p\}$ and has order p^{p+1}.

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Transitive groups of prime degree

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Let p be prime and $G \leq S_p$ be transitive. Then either G has a unique normal Sylow p-subgroup or G is doubly-transitive.

Corollary 50

Let $\mathrm{Cay}(\mathbb{Z}_p, S)$ *and* $\mathrm{Cay}(\mathbb{Z}_p, T)$ *be isomorphic circulant digraphs of prime order* p *that are neither complete graphs or complements of complete graphs.*

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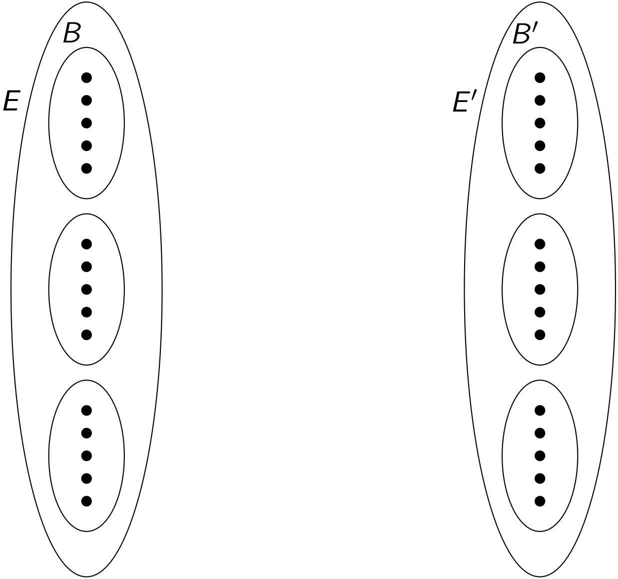
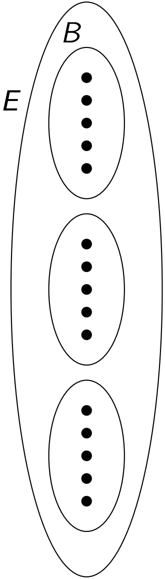

B
E
B'
E'



choose

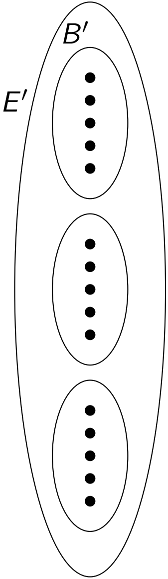
$\gamma \in \mathrm{fix}_G(\mathcal{B})$

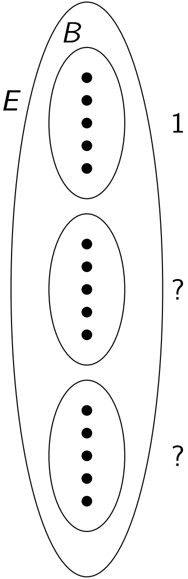
with

$\gamma|_{B'}$ a p-cycle

and

$\gamma|_B = 1$





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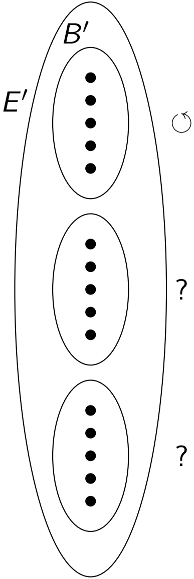

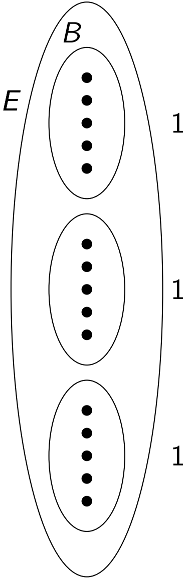

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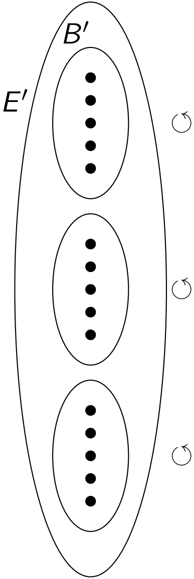
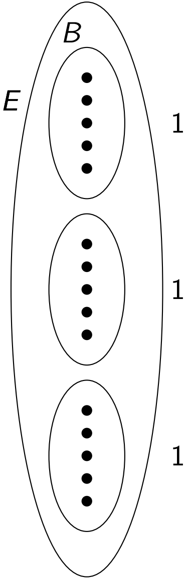
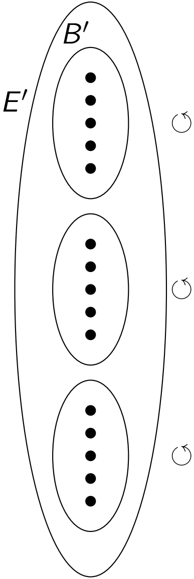
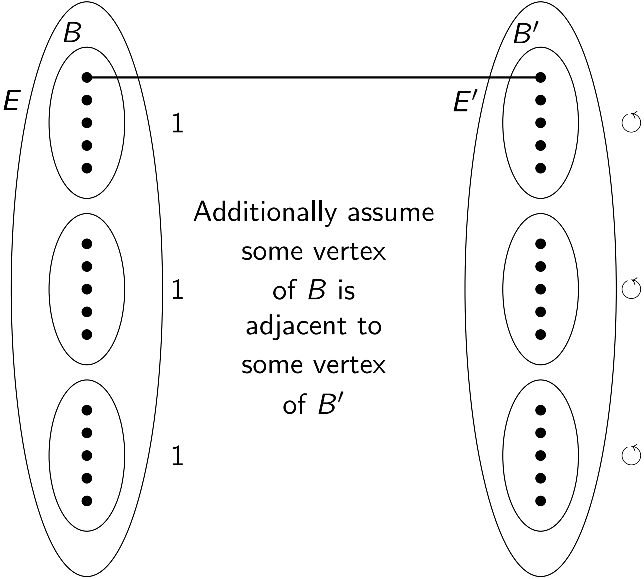


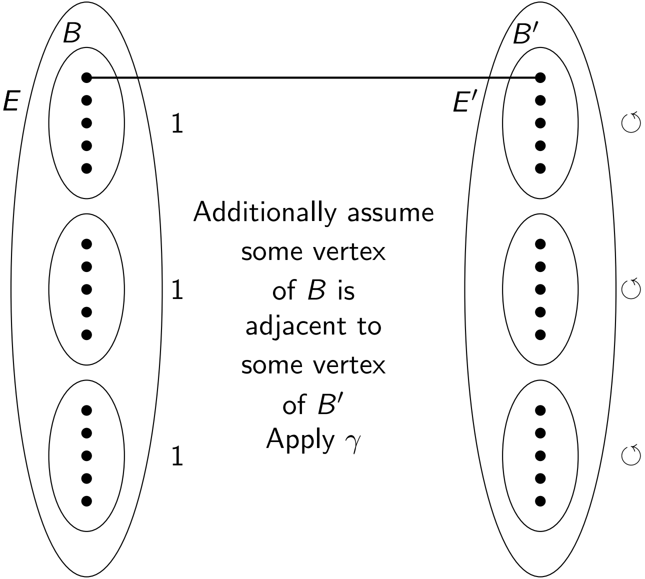


Additionally assume some vertex of B is adjacent to some vertex of B'



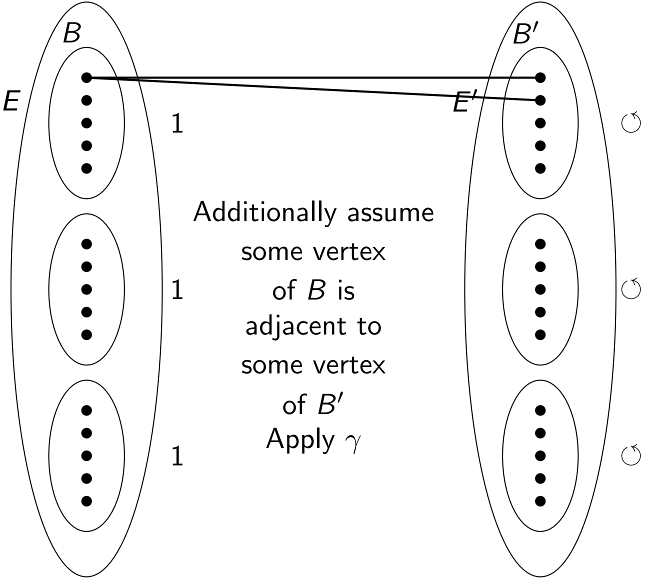


Additionally assume some vertex of B is adjacent to some vertex of B'



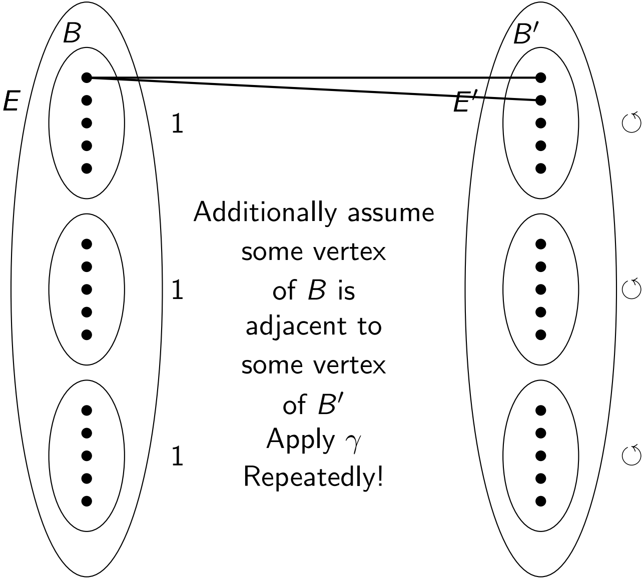
Additionally assume some vertex of B is adjacent to some vertex of B'

Apply γ



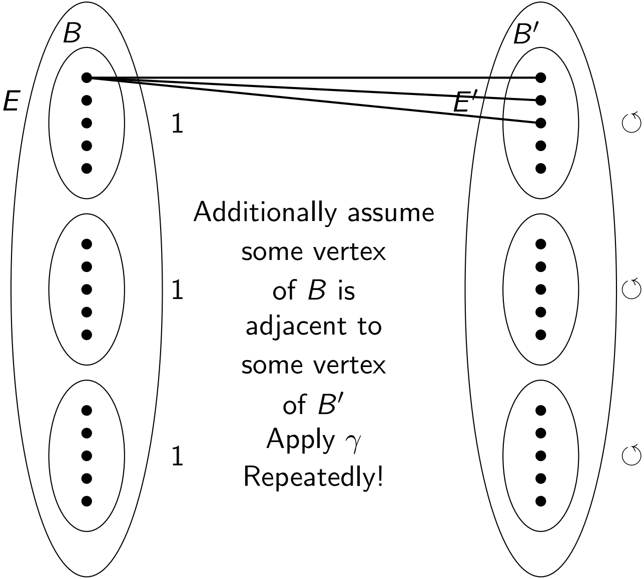
Additionally assume some vertex of B is adjacent to some vertex of B'

Apply γ



Additionally assume some vertex of B is adjacent to some vertex of B'

Apply γ Repeatedly!



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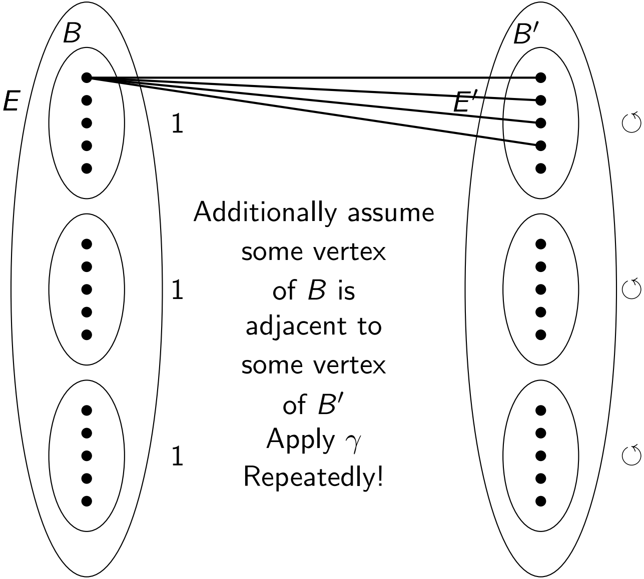
Apply γ Repeatedly!



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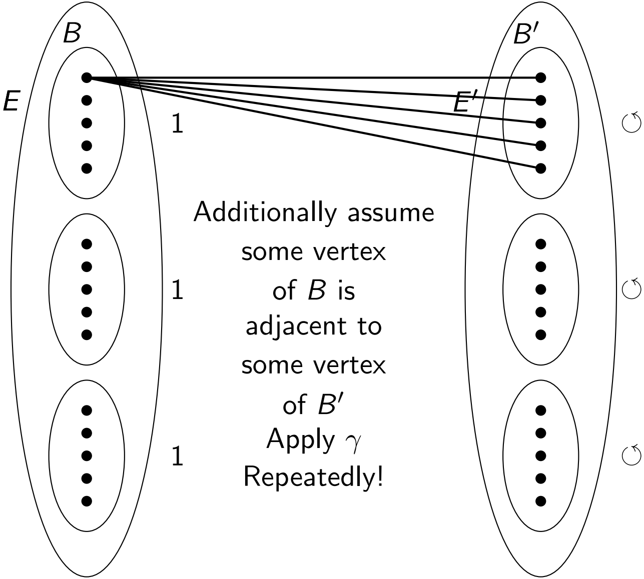
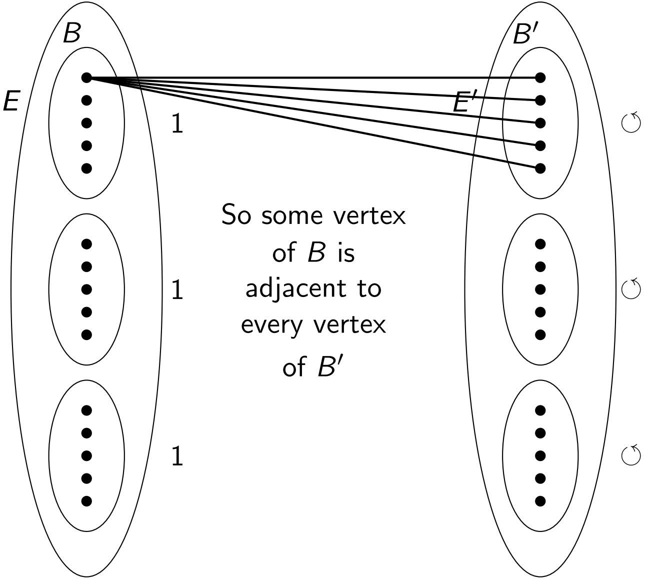
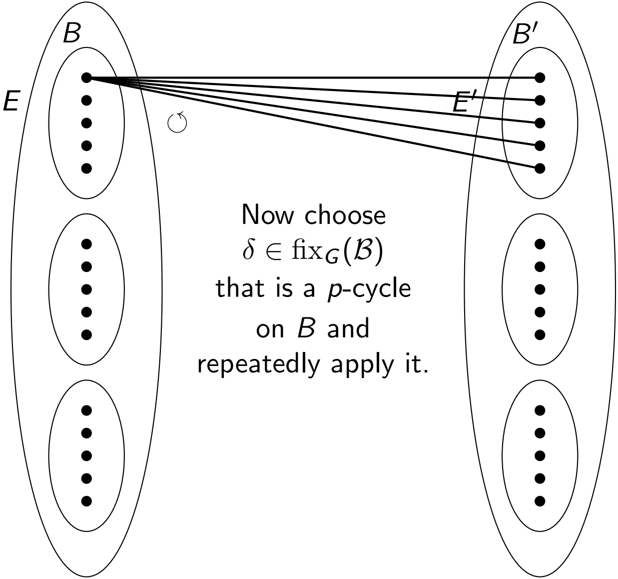


Additionally assume some vertex of B is adjacent to some vertex of B'

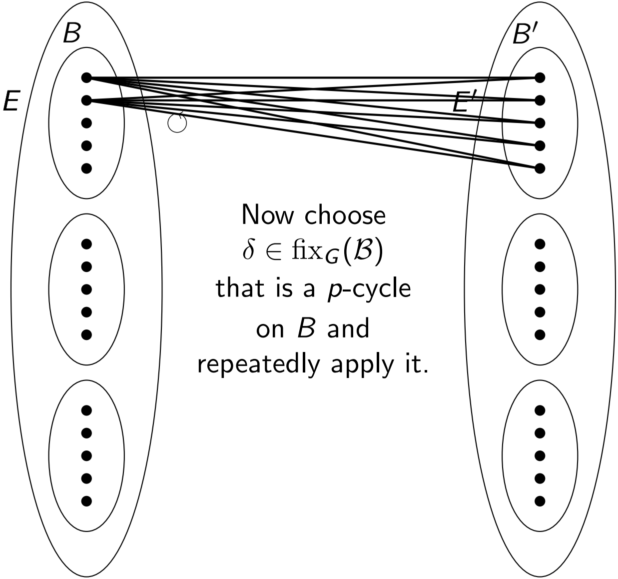
Apply γ Repeatedly!



So some vertex of B is adjacent to every vertex of B'



Now choose
$\delta \in \mathrm{fix}_G(\mathcal{B})$
that is a p-cycle
on B and
repeatedly apply it.



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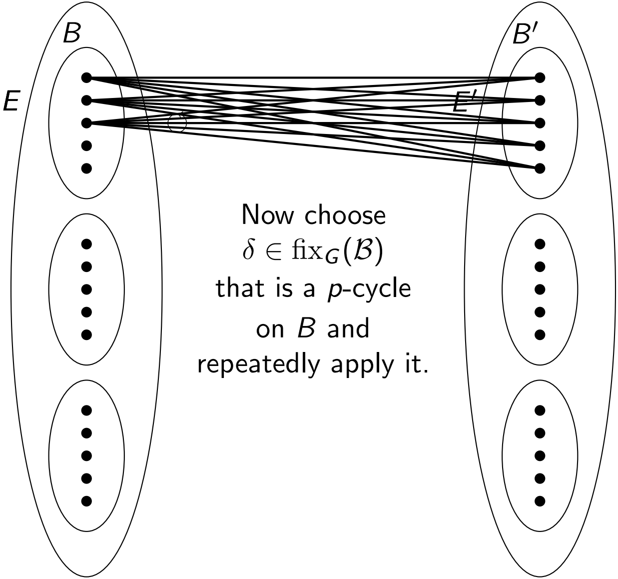

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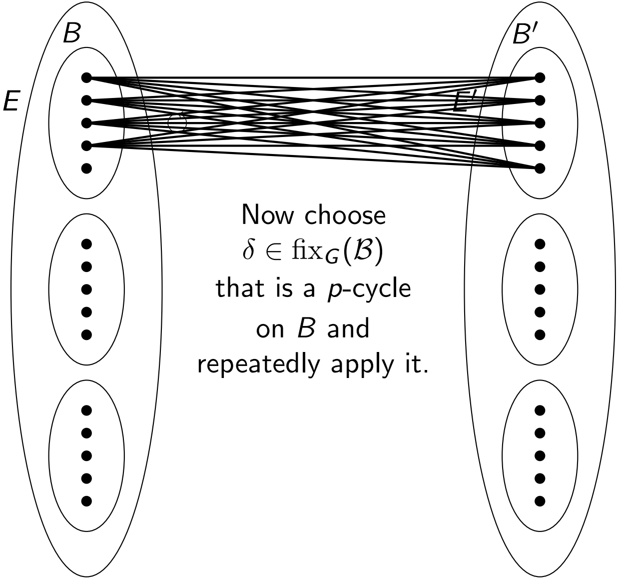

Now choose
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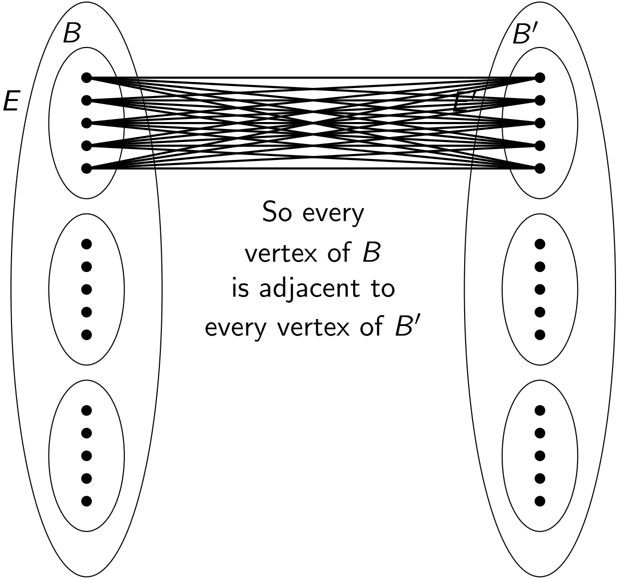
on B and
repeatedly apply it.



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Proof: If $\mathcal{E}$ consists of one block of size qp, then we first claim the kernel K of the restriction homomorphism from $\mathrm{fix}_G(\mathcal{B})$ into S_B, $B \in \mathcal{B}$, must be trivial.

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Lemma 54

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We may similarly assume that $P_1 \leq \mathrm{Aut}(\delta(\Gamma))$.

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Now, as the map ω defined by $\omega(i,j) = (i-a, j-b_i)$ is in $P_1 \leq \mathrm{Aut}(\Gamma)$, replacing δ with $\delta\omega$ we may assume that $\delta(i,j) = (mi, n_i j)$.

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Now, as the map ω defined by $\omega(i,j) = (i - a, j - b_i)$ is in $P_1 \leq \mathrm{Aut}(\Gamma)$, replacing δ with $\delta\omega$ we may assume that $\delta(i,j) = (mi, n_i j)$. As the map $(i,j) \mapsto (m^{-1}i, j)$ is an automorphism of $\mathbb{Z}_p^2$, replacing $\delta(\Gamma)$ by its image under this map by Lemma 53 we may also assume that $m = 1$ and $\delta(i,j) = (i, n_i j)$. Then $\delta^{-1} = (i, n_i^{-1} j)$, and letting $\tau : \mathbb{Z}_p^2 \mapsto \mathbb{Z}_p^2$ by $\tau(i,j) = (i+1, j)$ (so $\tau \in (\mathbb{Z}_p^2)_L \leq \mathrm{Aut}(\Gamma)$) we have

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\tau^{-1}\delta^{-1}\tau\delta(i,j) &= \tau^{-1}\delta^{-1}\tau(i, n_i j) \\
&= \tau^{-1}\delta^{-1}(i+1, n_i j) \\
&= \tau^{-1}(i+1, n_i n_{i+1}^{-1} j) \\
&= (i, n_i n_{i+1}^{-1} j)
\end{aligned}$$

As $\tau, \delta^{-1}\tau\delta \in P_1$, $\tau^{-1}\delta^{-1}\tau\delta \in P_1$. As $|\mathbb{Z}_p^*| = p - 1$, $|\tau^{-1}\delta^{-1}\tau\delta|$ is relatively prime to p and $\tau^{-1}\delta^{-1}\tau\delta = 1$. Thus $n_i n_{i+1}^{-1} = 1$ and $n_i = n_{i+1}$. Then $\delta \in \mathrm{Aut}(\mathbb{Z}_p^2)$ and $\mathbb{Z}_p^2$ is a CI-group with respect to digraphs.

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and so $\tau^p(i,j) = (i, j+1)$ and τ has order p^2. The argument is a little easier as $\langle\tau\rangle$ has a unique subgroup of order p, and so exactly one complete block system with blocks of size p instead of $p+1$ for $\mathbb{Z}_p^2$.

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Theorem 56

An abelian group with a cyclic Sylow subgroup is a Burnside group.

Let $\mathrm{Cay}(\mathbb{Z}_{qp}, S)$ be a Cayley digraph of $\mathbb{Z}_{qp}$, and $\delta \in S_{qp}$ such that $\delta^{-1}(\mathbb{Z}_{qp})_L\delta \leq \mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_{qp}, S))$.

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Next, we show that after appropriate conjugations we may assume that $\delta(\mathcal{B}) = \mathcal{B}$. This works pretty much as in the the argument for $\mathbb{Z}_p^2$, but is easier as $(\mathbb{Z}_{qp})_L$ contains a unique complete block system with blocks of size p. So we will not repeat this argument. This gives $\delta \in S_q \wr S_p$, and again similar arguments to the $\mathbb{Z}_p^2$ argument give that after appropriate conjugations we may assume that $\delta \in \mathrm{AGL}(1, q) \wr \mathrm{AGL}(1, p)$. Again we will not repeat this argument. This give $\delta(i, j) = (mi + a, n_i j + b_i)$ where $m \in \mathbb{Z}_q^*$, $b \in \mathbb{Z}_q$, $n_i \in \mathbb{Z}_p^*$, and $b_i \in \mathbb{Z}_p$. As the map $(i, j) \mapsto (i - a, j)$ is contained in $\mathrm{Aut}(\Gamma)$ we may assume that $a = 1$, Next, we show that after appropriate conjugations we may assume that $\delta(\mathcal{B}) = \mathcal{B}$. This works pretty much as in the the argument for $\mathbb{Z}_p^2$, but is easier as $(\mathbb{Z}_{qp})_L$ contains a unique complete block system with blocks of size p. So we will not repeat this argument. This gives $\delta \in S_q \wr S_p$, and again similar arguments to the $\mathbb{Z}_p^2$ argument give that after appropriate conjugations we may assume that $\delta \in \mathrm{AGL}(1, q) \wr \mathrm{AGL}(1, p)$. Again we will not repeat this argument. This give $\delta(i, j) = (mi + a, n_i j + b_i)$ where $m \in \mathbb{Z}_q^*$, $b \in \mathbb{Z}_q$, $n_i \in \mathbb{Z}_p^*$, and $b_i \in \mathbb{Z}_p$. As the map $(i, j) \mapsto (i - a, j)$ is contained in $\mathrm{Aut}(\Gamma)$ we may assume that $a = 1$, and as the map $(i, j) \mapsto (mi, j)$ is an automorphism of $\mathbb{Z}_{qp}$ we may assume that $m = 1$.

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Next, we show that after appropriate conjugations we may assume that $\delta(\mathcal{B}) = \mathcal{B}$. This works pretty much as in the the argument for $\mathbb{Z}_p^2$, but is easier as $(\mathbb{Z}_{qp})_L$ contains a unique complete block system with blocks of size p. So we will not repeat this argument. This gives $\delta \in S_q \wr S_p$, and again similar arguments to the $\mathbb{Z}_p^2$ argument give that after appropriate conjugations we may assume that $\delta \in \mathrm{AGL}(1,q) \wr \mathrm{AGL}(1,p)$. Again we will not repeat this argument. This give $\delta(i,j) = (mi + a, n_i j + b_i)$ where $m \in \mathbb{Z}_q^*$, $b \in \mathbb{Z}_q$, $n_i \in \mathbb{Z}_p^*$, and $b_i \in \mathbb{Z}_p$. As the map $(i,j) \mapsto (i-a,j)$ is contained in $\mathrm{Aut}(\Gamma)$ we may assume that $a = 1$, and as the map $(i,j) \mapsto (mi,j)$ is an automorphism of $\mathbb{Z}_{qp}$ we may assume that $m = 1$. Thus $\delta(i,j) = (i, n_i j + b_i)$, and $G/\mathcal{B} \cong \mathbb{Z}_q$. We will now deviate from our task and consider what happens if G has a Sylow q-subgroup of order q.

Lemma 57

If G has a Sylow q-subgroup of order q, then Γ is a CI-digraph with respect to $\mathbb{Z}_{qp}$.

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Proof.

If G has a Sylow q-subgroup of order q, then $(\mathbb{Z}_{qp})_L$ and $\delta^{-1}(\mathbb{Z}_{qp})_L\delta$ contain Sylow q-subgroups of G,

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If G has a Sylow q-subgroup of order q, then $(\mathbb{Z}_{qp})_L$ and $\delta^{-1}(\mathbb{Z}_{qp})_L\delta$ contain Sylow q-subgroups of G, so there exists $\gamma \in G$ such that a Sylow q-subgroup of $\gamma^{-1}\delta^{-1}(\mathbb{Z}_{qp})_L\delta\gamma$ is a Sylow q-subgroup of $(\mathbb{Z}_{qp})_L$.

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$$\delta(i, n_i^{-1}j - n_i b_i)$$

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$$\delta(i, n_i^{-1}j - n_i b_i) \quad = \quad (i, n_i(n_i^{-1}j - n_i^{-1}b_i) + b_i)$$

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$$\begin{aligned}\delta(i, n_i^{-1}j - n_i b_i) &= (i, n_i(n_i^{-1}j - n_i^{-1}b_i) + b_i)\\ &= (i, j - b_i + b_i)\end{aligned}$$

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$$\begin{aligned}
\delta(i, n_i^{-1}j - n_ib_i) &= (i, n_i(n_i^{-1}j - n_i^{-1}b_i) + b_i) \\
&= (i, j - b_i + b_i) \\
&= (i, j)
\end{aligned}$$

Then

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$$\tau^{-1}\delta^{-1}\tau\delta(i,j)$$

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$$\begin{aligned}\tau^{-1}\delta^{-1}\tau\delta(i,j) &= \tau^{-1}\delta^{-1}\tau(i, n_i j + b_i)\\ &= \tau^{-1}\delta^{-1}(i+1, n_i j + b_i)\end{aligned}$$

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\tau^{-1}\delta^{-1}\tau\delta(i,j) &= \tau^{-1}\delta^{-1}\tau(i, n_i j + b_i) \\
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&= \tau^{-1}(i+1, n_{i+1}^{-1}(n_i j + b_i) - n_{i+1}^{-1} b_{i+1}) \\
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Also, $\tau^{-1}\delta^{-1}\tau\delta \in \langle \tau \rangle$ we see that $\tau^{-1}\delta^{-1}\tau\delta = 1$.

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$$
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Then

$$\begin{aligned}
\tau^{-1}\delta^{-1}\tau\delta(i,j) &= \tau^{-1}\delta^{-1}\tau(i, n_i j + b_i)\\
&= \tau^{-1}\delta^{-1}(i+1, n_i j + b_i)\\
&= \tau^{-1}(i+1, n_{i+1}^{-1}(n_i j + b_i) - n_{i+1}^{-1} b_{i+1})\\
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$$\begin{aligned}
\tau^{-1}\delta^{-1}\tau\delta(i,j) &= \tau^{-1}\delta^{-1}\tau(i, n_i j + b_i)\\
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Then

$$\begin{aligned}
\tau^{-1}\delta^{-1}\tau\delta(i,j) &= \tau^{-1}\delta^{-1}\tau(i, n_i j + b_i) \\
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&= \tau^{-1}(i+1, n_{i+1}^{-1}(n_i j + b_i) - n_{i+1}^{-1} b_{i+1}) \\
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&= (i, n_{i+1}^{-1} n_i j + n_{i+1}^{-1}(b_i - b_{i+1})).
\end{aligned}$$

Also, $\tau^{-1}\delta^{-1}\tau\delta \in \langle\tau\rangle$ we see that $\tau^{-1}\delta^{-1}\tau\delta = 1$. Hence $n_{i+1}^{-1} n_i = 1$ and $n_i = n_{i+1}$ for all $i \in \mathbb{Z}_p$. Thus $n_i = n_j$ for all $i, j \in \mathbb{Z}_p$. Set $n_i = n$. Then $n^{-1}(b_i - b_{i+1}) = 0$ and $b_i = b_{i+1}$ for all $i \in \mathbb{Z}_p$. Then $b_i = b$ for all $i \in \mathbb{Z}_p$, and $\delta(i,j) = (i, nj + b)$. We may assume $b = 0$, and so $\delta \in \mathrm{Aut}(\mathbb{Z}_{qp})$ and the result follows.

Lemma 58

If $q \nmid (p-1)$ then a Sylow q-subgroup of G has order q.

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Proof.

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If $q \not| (p-1)$ then a Sylow q-subgroup of G has order q.

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Corollary 59

If $q \not| (p-1)$ then $\mathbb{Z}_{qp}$ is a CI-group with respect to digraphs.

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Corollary 59

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Theorem 60 (Pálfy 1987 [32])

If $q \not| (p-1)$ *then* $\mathbb{Z}_{qp}$ *is a CI-group.*

We now return from our detour and no longer assume that $q \nmid (p-1)$.

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Otherwise, we have that $(\mathbb{Z}_q)_L \wr (\mathbb{Z}_p)_L \leq \mathrm{Aut}(\Gamma)$ (and is in $\mathrm{Aut}(\delta(\Gamma))$), and so the map $(i,j) \mapsto (i, j - b_i)$ is contained in $\mathrm{Aut}(\Gamma)$,

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So we may assume without loss of generality that $\ell_0 = 1$ (but that ℓ_{p-1} has changed and all other ℓ_i are the same). Repeat the above conjugation using block B_{p-1} we will see that we may assume that $\ell_{p-1} = 0$, ℓ_{p-2} has changed and $\ell_0 = 1$ (as long as $q \neq 2$). We repeat this procedure for blocks $B_0, B_{p-1}, \ldots, B_2$ (so in total $p-1$ times) and we may assume that

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Now, the conjugations of course do not change the order of τ', and so $\Pi_{i=0}^{q-1}\ell_i = 1$. Thus $\ell_1 = 1$ and $\tau' = \tau$. The only way this can occur (and we have seen the computation) is if $n_0 = n_1 = \ldots = n_{q-1}$ in which case $\delta \in \mathrm{Aut}(\mathbb{Z}_{qp})$. Then Γ is a CI-digraph of $\mathbb{Z}_{qp}$ and the result follows.

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Corollary 62

If $\mathbb{Z}_n$ is a CI-group with respect to digraphs then $n = m, 2m$, or $4m$ where m is odd and square-free.

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Theorem 64 (Muzychuk 1995 [28], 1997 [29])

$\mathbb{Z}_n$ is a CI-group with respect to (di)graphs if and only if $n = m, 2n, 4m$, and additionally in the case of graphs for $n = 8, 9$ and 18, where m is odd and square-free.

The Solution to the Isomorphism Problem for Circulants

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More is known about circulants! Evdokimov and Ponomarenko in 2003 [17] solved the recognition problem for circulants - that is, given graph they have an algorithm to determine if the graph is isomorphic to a circulant graph. As consequences, they also solve the isomorphism problem and Ponomarenko [33] gave a polynomial time algorithm to compute the full automorphism group of a circulant.

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Theorem 67

A CI-group with respect to graphs is solvable.

All Possible CI-groups with respect to graphs

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The next result is a combination of a result of Li, Lu, and Pálfy 2007 [24] and Somlai 2011 [36].

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1. *If G does not contain elements of order 8 or 9, then $G = H_1 \times H_2 \times H_3$, where the orders of H_1, H_2, and H_3 are pairwise relatively prime, and*
   1.1 *H_1 is an abelian group, and each Sylow p-subgroup of H_1 is isomorphic to $\mathbb{Z}_p^k$ for $k < 2p + 3$ or $\mathbb{Z}_4$;*
   1.2 *H_2 is isomorphic to one of the groups $E(2, M)$, $E(4, M)$, Q_8, or 1;*
   1.3 *H_3 is isomorphic to one of the groups $E(3, M)$, A_4 , or 1.*
2. *If G contains elements of order 8, then $G \cong E(8, M)$ or $\mathbb{Z}_8$.*

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Which of these groups are CI-groups with respect to graphs?

- $\mathbb{Z}_n$, where $n \in \{8, 9, 18, k, 2k, 4k\}$ and k is odd and square-free Muzychuk 1995, 1997 [28, 29];

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The isomorphism problem for all combinatorial objects

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G is a CI-group if and only if $|G| = 4$ *or* n *where* $\gcd(n, \varphi(n)) = 1$*, where* φ *is Euler's phi function.*

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Let $m = p_1 \cdots p_r$ *where each* p_i *is prime,* $\gcd(m, \varphi(m)) = 1$*, and* $n = p_1^{a_1} \cdots p_r^{a_r}$*.*

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Conjecture 75 (Folklore)

If G and H are CI-groups with respect to digraphs of relatively prime order then $G \times H$ is a CI-group with respect to digraphs.

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1. *Classify the digraphs in terms of a minimal transitive subgroup of their automorphism group.*
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1. *Classify the digraphs in terms of a minimal transitive subgroup of their automorphism group.*
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More information about the Cayley isomorphism problem can be found in the still somewhat recent survey of C.H. Li [23].

Thanks!

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